A Note: Conditions For The Equivalence Of The Autoregressive Latent Trajectory Model (ALT) And A Latent Growth Curve Model With Autoregressive Disturbances (LGCWAD) With Finite Time

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Abstract

This paper extends the conditions for the equivalence of the autoregressive latent trajectory model and a latent growth curve model with autoregressive disturbances and presented the conditions when the time is finite.

Keywords: Autoregressive Latent Trajectory Model, Latent Growth Curve Model With Autoregressive Disturbances, Conditions for the Equivalence

Introduction

There are two historically important structural equation modeling (SEM) models for the analysis of longitudinal panel data: the autoregressive (simplex) model and the latent growth curve model. By combining the two models, Bollen et Curran (2004) proposed the autoregressive latent trajectory model (ALT). In ALT model, the autoregressive relationships are modeled within observed variables. Another related model is the latent growth curve model with autoregressive disturbances (LGCWAD) (e.g., Diggle et. al 1994). In LGCWAD model, the autoregressive relationships are modeled within the disturbances.

Regarding the equivalence of ALT model and LGCWAD model, there are two perspectives in literature. Bollen et Curran (2004) stated that they are two different models and can't be equivalent. While Hamaker (2005) stated it can be shown that the two models are algebraically equivalent when the autoregressive parameter ρ in the ALT model is invariant over time and lies between -1 and 1. She investigated the equivalency of the two models with both formula derivation and numeric examples.

However, Hamaker (2005) only consider the case where the time in models goes to infinity. When the time is finite, the conditions under which the two models are equivalent with infinite time no longer hold. In practice, time is usually finite and the influence of the initial value is important. We need to find the new conditions for the equivalence of ALT model and LGCWAD model. This paper obtained the new conditions, and also shows the conditions with finite time are more general than those with infinite time because the latter can be derived from the former. In addition to the formula derivation, numeric examples are also provided in this paper.

Descriptions of Two Models

Model 1: The ALT Model

When time variable is from $-\infty$ to $+\infty$, an ALT model can be expressed as:

$$y_{it} = \alpha_i + t\,\beta_i + \rho\, y_{i(t-1)} + e_{it} \tag{1}$$
$$\alpha_i = \mu_\alpha + \zeta_{\alpha i} \tag{2}$$
$$\beta_i = \mu_\beta + \zeta_{\beta i} \tag{3}$$

where $t \in (-\infty, +\infty)$, $E(e_{it}) = 0$, $COV(e_{it}, y_{i,t-1}) = 0$, $COV(e_{it}, \beta_i) = 0$, $COV(e_{it}, \alpha_i) = 0$, $E(e_{it}e_{jt}) = 0$, $COV(e_{it}, e_{it}) = \sigma^2_{et}$, $COV(e_{it}, e_{i,t+k}) = 0$. $\zeta_{\alpha i}$ and $\zeta_{\beta i}$ are two residual terms with means of zero and we allow them to correlate. But they are uncorrelated with e_{it}. The path diagram for ALT model with infinite time is illustrated in Figure (1).

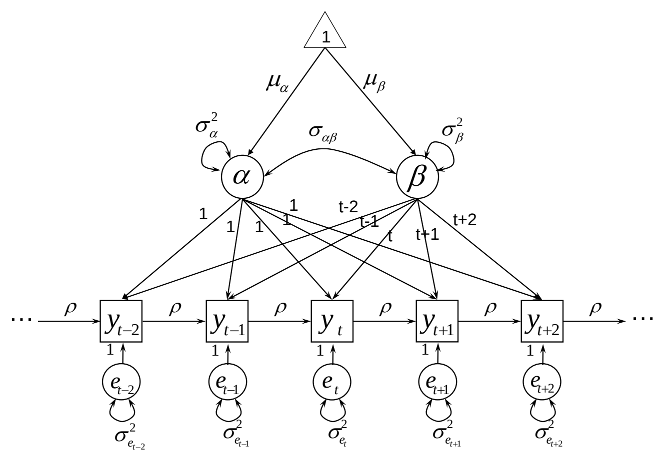

Figure 1. The Path Diagram for ALT Model With Infinite Time For 5 Waves

But when time variable is not infinite and starts from timepoint 1, a specific form of ALT considering the initial value y_{i1} is:

$$y_{i1} = \upsilon_1 + e_{i1} \tag{4}$$
$$y_{it} = \alpha_i + t\,\beta_i + \rho\, y_{i(t-1)} + e_{it} \qquad (t \geq 2) \tag{5}$$
$$\alpha_i = \mu_\alpha + \zeta_{\alpha i} \tag{6}$$
$$\beta_i = \mu_\beta + \zeta_{\beta i} \tag{7}$$

where $t = 2, 3, ..., T$, $E(e_{it}) = 0$, $COV(e_{it}, y_{i,t-1}) = 0$, $COV(e_{it}, \beta_i) = 0$, $COV(e_{it}, \alpha_i) = 0$, $E(e_{it}e_{jt}) = 0$, $COV(e_{it}, e_{it}) = \sigma^2_{et}$, $COV(e_{it}, e_{i,t+k}) = 0$. $\zeta_{\alpha i}$ and $\zeta_{\beta i}$ are two residual terms with means of zero and we allow them to correlate. But they are uncorrelated with e_{it}. The predetermined $y_{i,1}$ correlates with α_i and β_i. The path diagram for ALT model with the first 5 waves is illustrated in Figure (2).

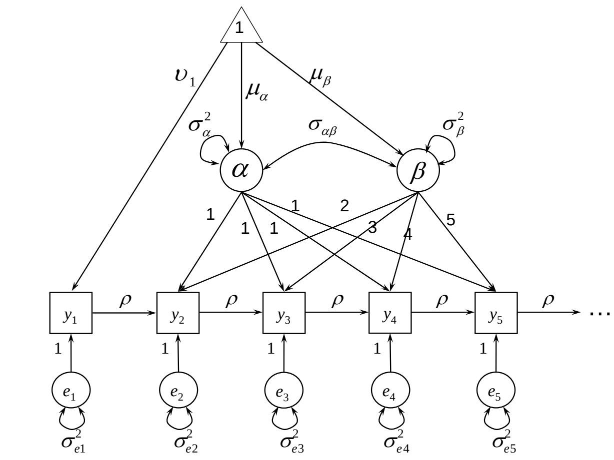

Figure 2. The Path Diagram For ALT Model With Finite Time For 5 Waves

Model 2: The LGCWAD Model

When time variable is from $-\infty$ to $+\infty$, an ALT model can be expressed as:

$$\begin{aligned} y_{it} &= \delta_i + t\gamma_i + z_{it} \\ \delta_i &= \mu_\delta + \zeta_{\delta i} \\ \gamma_i &= \mu_\gamma + \zeta_{\gamma i} \\ z_{it} &= \rho z_{i,t-1} + e_{it} \end{aligned} \tag{8}$$

where e_{it} has a mean of zero and is uncorrelated with $e_{i,t-1}$, δ_i and γ_i. The path diagram for the corresponding LGCWAD model with infinite time is illustrated in Figure (3).

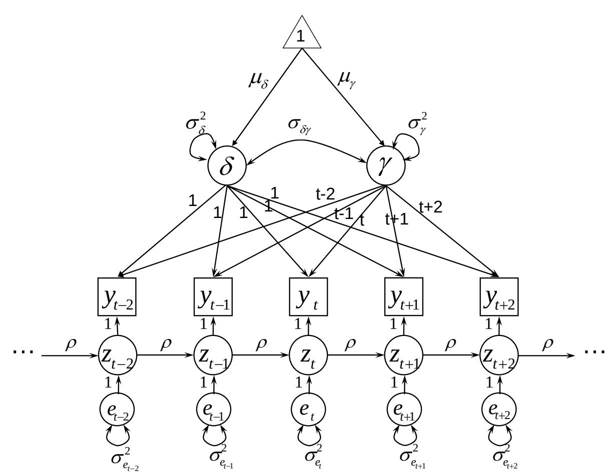

Figure 3. The Path Diagram for LGCWAD Model With Infinite Time For 5 Waves

Similarly, when time is finite, the model of LGCWAD Model considering the initial value y_{i1} can be expressed as

$$\begin{aligned} y_{i1} &= \upsilon_1 + e_{i1} \\ y_{it} &= \delta_i + t\gamma_i + z_{it} \qquad (t \geq 2) \\ \delta_i &= \mu_\delta + \zeta_{\delta i} \\ \gamma_i &= \mu_\gamma + \zeta_{\gamma i} \\ z_{it} &= \rho z_{i,t-1} + e_{it} \end{aligned} \tag{9}$$

where e_{it} has a mean of zero and is uncorrelated with $e_{i,t-1}$, δ_i and γ_i. The path diagram for LGCWAD model with finite time for 5 waves is illustrated in Figure (4).

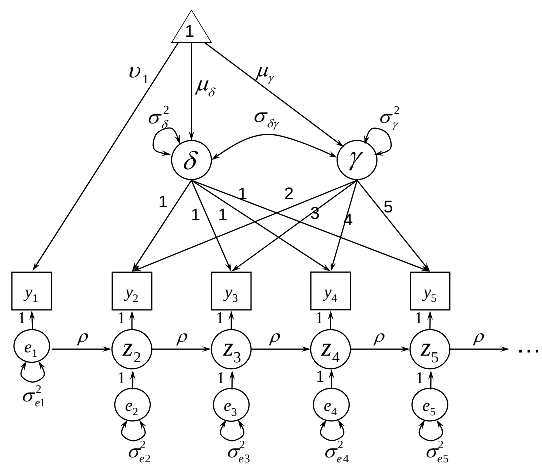

Figure 4. The Path Diagram For LGCWAD Model With Finite Time For 5 Waves

The Conditions for the Equivalence of Two Models

When the time is infinite and the autoregressive parameter ρ in ALT is invariant over time, Hamaker (2005) showed the conditions under which the two models are algebraically equivalent. She re-expressed the ALT model with

$$\begin{aligned} y_{it} = &\left[\frac{1}{1-\rho}\alpha_i - \frac{\rho}{(1-\rho)^2}\beta_i\right] + t\left(\frac{\beta_i}{1-\rho}\right) \\ &+ \left[\rho^{t-1}e_{i1} + \rho^{t-2}e_{i2} + ... + \rho^2 e_{i(t-2)} + \rho\, e_{i(t-1)} + e_{it}\right], \end{aligned} \tag{10}$$

and the conditions making the two model equivalent are

$$\delta_i = \frac{1}{1-\rho}\alpha_i - \frac{\rho}{(1-\rho)^2}\beta_i \tag{11}$$

$$\gamma_i = \frac{\beta_i}{1-\rho} \tag{12}$$

$$z_{it} = \rho^{t-1}e_{i1} + \rho^{t-2}e_{i2} + ... + \rho^2 e_{i(t-2)} + \rho\, e_{i(t-1)} + e_{it} \tag{13}$$

But when the time is finite, we need to re-investigate those conditions.

Equivalence Conditions

We start from the Eq. (5) in the ALT model and use Eq.(4). For individual i, we have

$$\begin{aligned}
y_{it} &= \alpha_i + t\,\beta_i + \rho\, y_{i(t-1)} + e_{it} \qquad (t \geq 2)\\
&= \alpha_i + t\,\beta_i + \rho\,[\alpha_i + (t-1)\,\beta_i + \rho\, y_{i(t-2)} + e_{i(t-1)}] + e_{it}\\
&= (1+\rho)\alpha_i + [t + \rho(t-1)]\beta_i + \rho^2 y_{i(t-2)} + [\rho e_{i(t-1)} + e_{it}]\\
&= (1+\rho)\alpha_i + [t + \rho(t-1)]\beta_i + \rho^2[\alpha_i + (t-2)\,\beta_i + \rho\, y_{i(t-3)} + e_{i(t-2)}] + [\rho e_{i(t-1)} + e_{it}]\\
&= (1+\rho+\rho^2)\alpha_i + [t + \rho(t-1) + \rho^2(t-2)]\beta_i + \rho^3 y_{i(t-3)} + [\rho^2 e_{i(t-2)} + \rho e_{i(t-1)} + e_{it}]\\
&= \ldots\\
&= (1+\rho+\rho^2+\ldots+\rho^{t-2})\alpha_i + [t + \rho(t-1) + \rho^2(t-2) + \ldots + \rho^{t-2}\,2]\beta_i\\
&\quad +\rho^{t-1} y_{i1} + [\rho^{t-2}e_{i2} + \ldots + \rho^2 e_{i(t-2)} + \rho e_{i(t-1)} + e_{it}]\\
&= (1+\rho+\rho^2+\ldots+\rho^{t-2})\alpha_i + \{t(1+\rho+\rho^2+\ldots+\rho^{t-2}) - \rho[1+2\rho+\ldots+(t-2)\rho^{t-3}]\}\beta_i\\
&\quad +\rho^{t-1}\upsilon_1 + [\rho^{t-1}e_{i1} + \rho^{t-2}e_{i2} + \ldots + \rho^2 e_{i(t-2)} + \rho e_{i(t-1)} + e_{it}]\\
&\triangleq A\,\alpha_i + (tA - \rho C)\beta_i + \rho^{t-1}\upsilon_1 + z_{it}. \qquad (14)
\end{aligned}$$

where $\triangleq$ means "is defined as", and $A = 1+\rho+\rho^2+\ldots+\rho^{t-2}$, $C = 1+2\rho+\ldots+(t-2)\rho^{t-3}$, $z_{it} = \rho^{t-1}e_{i1}+\rho^{t-2}e_{i2}+\ldots+\rho^2 e_{i(t-2)}+\rho\, e_{i(t-1)}+e_{it}$. Notice that by this definition, we have $z_{it} = \rho\, z_{i(t-1)} + e_{it}$ as in Eq. (20). Also notice that when $|\rho| < 1$, we have

$$\begin{aligned}
A &= \frac{1-\rho^{t-1}}{1-\rho},\\
C &= \frac{d}{d\rho}(1+\rho+\rho^2+\ldots+\rho^{t-2})\\
&= \frac{d}{d\rho}(\frac{1-\rho^{t-1}}{1-\rho})\\
&= \frac{1-\rho^{t-1}}{(1-\rho)^2} - \frac{(t-1)\rho^{t-2}}{1-\rho},
\end{aligned}$$

then $tA - \rho C = \frac{t-\rho^{t-1}}{1-\rho} - \rho\frac{1-\rho^{t-1}}{(1-\rho)^2}$. Replacing A and $tA - \rho C$ in Eq. (14), we have

$$y_{it} = \left\{\frac{1-\rho^{t-1}}{1-\rho}\alpha_i - \left[\frac{\rho^{t-1}}{1-\rho} + \rho\frac{1-\rho^{t-1}}{(1-\rho)^2}\right]\beta_i + \rho^{t-1}\upsilon_1\right\} + t\left(\frac{\beta_i}{1-\rho}\right) + z_{it} \qquad (15)$$

Because of Eq. (6) and (7), then Eq. (15) is re-expressed a LGCWAD model as follows.

$$y_{i1} = \upsilon_1 + e_{i1} \qquad (16)$$

$$y_{it} = \delta_{it} + t\gamma_i + z_{it} \qquad (17)$$

$$\delta_{it} = \mu_{\delta_t} + \zeta_{\delta_{it}} \qquad (18)$$

$$\gamma_i = \mu_\gamma + \zeta_{\gamma i} \qquad (19)$$

$$z_{it} = \rho z_{i,t-1} + e_{it} \qquad (20)$$

where

$$\delta_{it} = \frac{1-\rho^{t-1}}{1-\rho}\alpha_i - \left[\frac{\rho^{t-1}}{1-\rho} + \rho\frac{1-\rho^{t-1}}{(1-\rho)^2}\right]\beta_i + \rho^{t-1}\upsilon_1 \qquad (21)$$

$$\gamma_i = \frac{\beta_i}{1-\rho} \qquad (22)$$

$$z_{it} = \rho^{t-1}e_{i1} + \rho^{t-2}e_{i2} + \ldots + \rho^2 e_{i(t-2)} + \rho\, e_{i(t-1)} + e_{it} \qquad (23)$$

Notice that in (16) the intercept δ_{it} is a time-varying variable. For different t values, values of δ_{it} are different. In more detail, the equivalence conditions are

$$\mu_{\delta_t} = \frac{1-\rho^{t-1}}{1-\rho}\mu_\alpha - \left[\frac{\rho^{t-1}}{1-\rho} + \rho\frac{1-\rho^{t-1}}{(1-\rho)^2}\right]\mu_\beta + \rho^{t-1}\upsilon_1 \quad (24)$$

$$\zeta_{\delta_{it}} = \frac{1-\rho^{t-1}}{1-\rho}\zeta_{\alpha_i} - \left[\frac{\rho^{t-1}}{1-\rho} + \rho\frac{1-\rho^{t-1}}{(1-\rho)^2}\right]\zeta_{\beta_i} \quad (25)$$

$$\mu_\gamma = \frac{\mu_\beta}{1-\rho} \quad (26)$$

$$\zeta_{\gamma_i} = \frac{\zeta_{\beta_i}}{1-\rho} \quad (27)$$

The Relationship Between Two Sets of Conditions

When t goes to infinity, ρ^t goes to zero since $|\rho| < 1$. In this case we can use (11) to replace (21). So, (10) is a special case of (15) when time goes to infinity.

In practice, when the time t is large enough, we usually approximate ρ^t with zero.

A Numeric Example

From the difference between (21) and (11), we see that the auto-regressive parameter ρ plays an important role. For different ρ values, the y_t values are different for the same t. We compared the y_t for different models with different ρ in Figure (5).

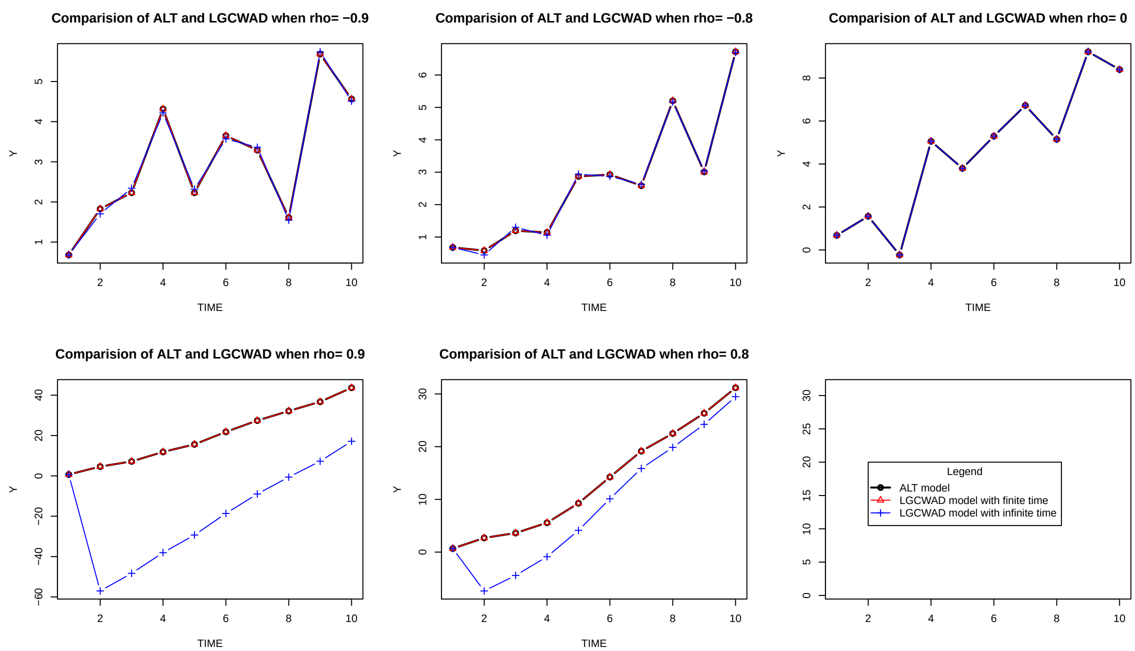

Figure 5. The comparison of ALT and LGCWAD model with different ρs.

Table 1: Numeric Example Showing the Equivalence Between the Two Models With $\rho = 0.8$

		LGCWAD Model					ALT Model
		Finite:Eq.(16)-(27)				Inf:Eq.(10)-(12)	Eq.(4)-(7)
t	e_t	z_t	μ_{δ_t}	ζ_{δ_t}	y_t	y_t	y_t
1	-0.00887716	-0.00887716	0.6884196	0.00000000	0.6795425	0.6795425	0.6795425
2	0.95429497	0.94719325	-6.4492643	0.21971931	3.5267264	-6.5269756	3.5267264
3	0.98252165	1.74027625	-8.1594114	-0.08087397	6.7136081	-1.3293535	6.7136081
4	-0.35110670	1.04111429	-9.5275292	-0.32134859	8.8103929	2.3760236	8.8103929
5	0.37350119	1.20639262	-10.6220233	-0.51372828	12.0933365	6.9458410	12.0933365
6	-0.63055922	0.33455488	-11.4976187	-0.66763204	14.5965388	10.4785424	14.5965388
7	-0.33743414	-0.06979024	-12.1980949	-0.79075505	17.7731335	14.4787364	17.7731335
8	-0.38505853	-0.44089072	-12.7584759	-0.88925345	21.1476927	18.5121750	21.1476927
9	-0.90684363	-1.25955620	-13.2067808	-0.96805217	24.2064627	22.0980486	24.2064627
10	0.68694796	-0.32069700	-13.5654246	-1.03109115	29.1281782	27.4414469	29.1281782
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
29	1.95563084	1.90265104	-14.9793256	-1.27961312	113.3753461	113.3510378	113.3753461
30	-0.85808444	0.66403640	-14.9834605	-1.28033991	116.5364089	116.5169622	116.5364089

Note 1: In this example, $\mu_\alpha = 1$, $\mu_\beta = 1$, the random number $\zeta_\alpha = -0.7330181$ which was generated from $N(0,1)$, the random number $\zeta_\beta = -0.1190922$ which was generated from $N(0, 0.1)$. All these values are used in both models. $\mu_\gamma = 5$ which was calculated based on Eq.(26), $\zeta_\gamma = -0.5954609$ which was calculated based on Eq.(27). $v_1 = 0.6884196$ which was also a random number generated from $N(0,1)$ and used as the initial mean when $t = 1$ in both models.
Note 2: The first column indicates time. The second column contains the i.i.d. disturbance that were generated from $e_t \sim N(0,1)$. The first element of column 3 is set to be equal to e_1. All other elements in the third column (i.e., z_2 to z_{10}) were then obtained by use of Eq.(20). The fourth and the fifth columns are calculated by use of Eq.(24) and (25), respectively. Using Eq.(17) gave the value for y in column 6. The seventh column contains y based on Eq.(4) and Eq.(5).

Références

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